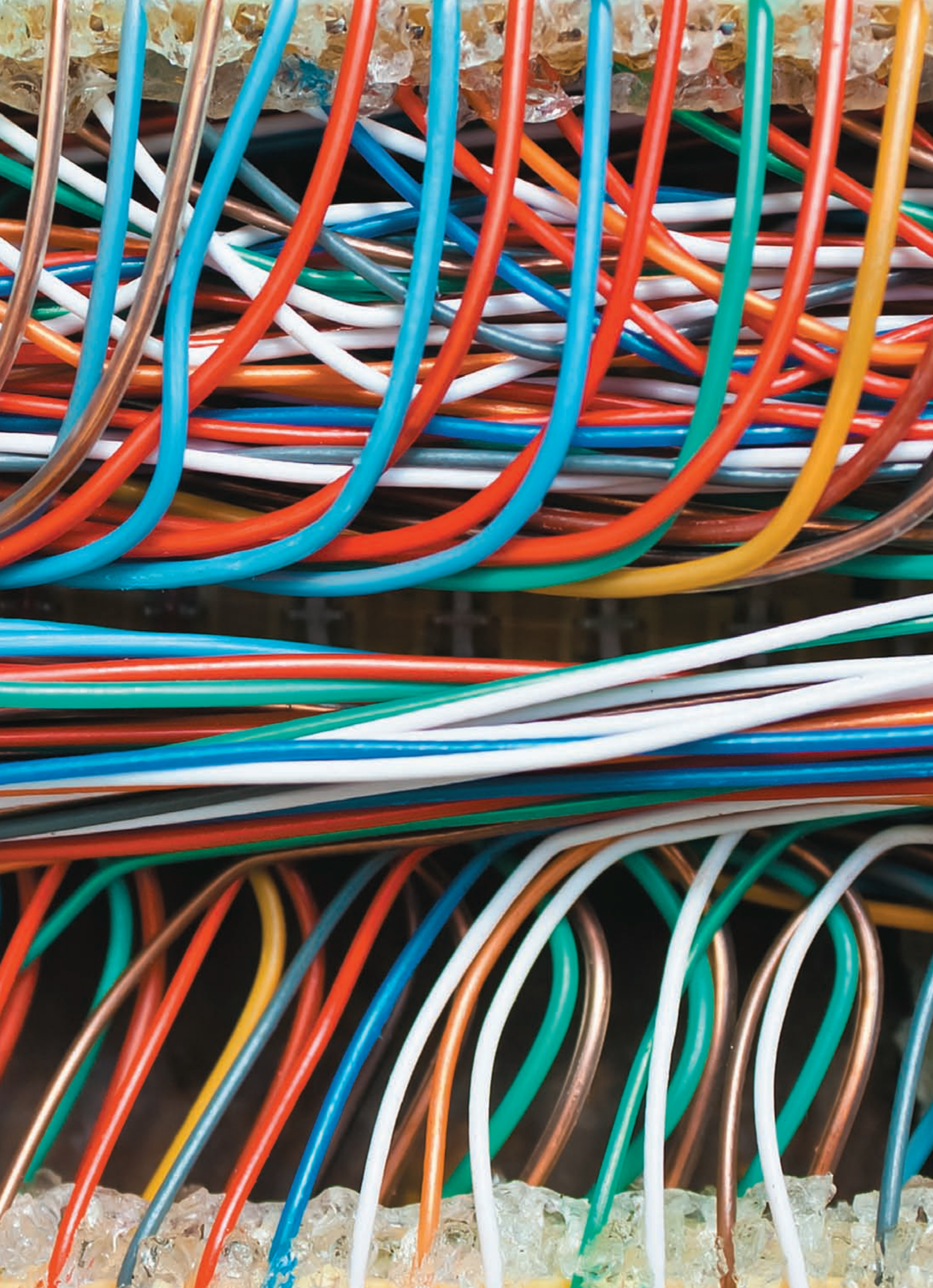


3

TERMINOLOGY

absolute value
composite function
inverse function
one-to-one function
rational function
reciprocal function

FUNCTIONS AND SKETCHING GRAPHS

FUNCTIONS AND SKETCHING GRAPHS

Prior learning

FUNCTIONS

- determine when the composition of two functions is defined (ACMSM092)
- find the composition of two functions (ACMSM093)
- determine if a function is one-to-one (ACMSM094)
- consider inverses of one-to-one functions (ACMSM095)
- examine the reflection property of the graph of a function and the graph of its inverse. (ACMSM096)

SKETCHING GRAPHS

- use and apply the notation $|x|$ for the absolute value for the real number x and the graph of $y = |x|$ (ACMSM098)
- examine the relationship between the graph of $y = f(x)$ and the graphs of $y = \frac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$ (ACMSM099)
- sketch the graphs of simple rational functions where the numerator and denominator are polynomials of low degree. (ACMSM100) AC

3.01 COMPOSITION OF FUNCTIONS

You can make more complicated functions from simple ones by combining them. You can make the output of one function the input of another. This is known as **composition** of functions. If you use $f(x) = x^2$ and $g(x) = x + 2$, then you get the function $g \circ f$ by doing $g[f(x)] = gf(x)$.

In this case, $g[f(x)] = g(x^2) = x^2 + 2$. Notice that $f[g(x)] = (x + 2)^2$, a different function.

IMPORTANT

The **composition** $g \circ f$ is the function given by $g[f(x)] = gf(x)$. This is also written as gf and is often called a 'function of a function'.

Example 1

Find $g[f(x)]$ given that:

a $f(x) = 2x$ and $g(x) = x^2 + 3$

b $f(x) = \sin(x)$ and $g(x) = \frac{1}{x}$

Solution

a Substitute $f(x) = 2x$.
Substitute $2x$ into $g(x)$.

$g[f(x)] = g(2x)$
$g[f(x)] = (2x)^2 + 3$
$g[f(x)] = 4x^2 + 3$

b Substitute $f(x) = \sin(x)$.
Substitute $\sin(x)$ into $g(x)$.

$g[f(x)] = g[\sin(x)]$
$g[f(x)] = \frac{1}{\sin(x)}$
$g[f(x)] = \operatorname{cosec}(x)$

Example 2

Find $f[g(x)]$, and state the domain given that:

a $f(x) = 2x$ and $g(x) = x^2 + 3$

b $f(x) = \sin(x)$ and $g(x) = \dfrac{1}{x}$

c $f(x) = \sqrt{x}$, $g(x) = 4 - x^2$

Compare your results for **a** and **b** with those in Example **1**.

Solution

a Substitute $g(x) = x^2 + 3$.

$f[g(x)] = f(x^2 + 3)$

Substitute $x^2 + 3$ into $f(x)$.

$f[g(x)] = 2(x^2 + 3)$

$f[g(x)] = 2x^2 + 6$

The function is valid for any value of x.

The domain is $x \in \boldsymbol{R}$.

Compare the results.

This is different to $g[f(x)] = 4x^2 + 3$.

b Substitute $g(x) = \dfrac{1}{x}$

$f[g(x)] = f\left(\dfrac{1}{x}\right)$

Substitute $\dfrac{1}{x}$ into $f(x)$.

$f[g(x)] = \sin\left(\dfrac{1}{x}\right)$

The function is not valid for 0.

The domain is $x \in \boldsymbol{R}/0$.

Compare the results.

This is different to $g[f(x)] = \text{cosec}\,(x)$.

c Substitute $g(x) = 4 - x^2$.

$f[g(x)] = f(4 - x^2)$

$= \sqrt{4 - x^2}$

Square roots of negatives are not real.

The domain is $-2 \le x \le 2$.

You can write many functions as the composition of simpler functions. For example, take the function, $h(x) = e^{3x}$. This function may be written as a **composite function** $h(x) = g[f(x)]$, where $g(x) = e^x$ and $f(x) = 3x$.

Example 3

Decompose the following functions into the form $g[f(x)]$, and state the domain and range for each.

a $h(x) = 5x - 2$

b $h(x) = e^{2x-1}$

c $h(x) = x^2 + 2x + 1$

Solution

a Write the operations involved as functions.

Try $f(x) = 5x$ and $g(x) = x - 2$

Try the composition $f \circ g$.

$f[g(x)] = f(x - 2)$

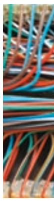
$= 5(x - 2)$

$= 5x - 10$

Do it the other way round as $g \circ f$.

$g[f(x)] = g(5x)$

$= 5x - 2$ **OK**



	Now write the domain.	x can be anything, so the domain is $x \in \boldsymbol{R}$.
	The graph is a straight line, so you can get every possible value of y.	y can be anything, so the range is $y = h(x) \in \boldsymbol{R}$.
b	Write the operations involved as functions.	$g(x) = e^x$, $f(x) = 2x - 1$
	Try the composition $f \circ g$.	$f[g(x)] = f(e^x)$ $f[g(x)] = 2e^x - 1$
	Do it the other way round as $g \circ f$.	$g[f(x)] = g(2x - 1)$ $g(2x - 1) = e^{2x-1}$ As required.
	Now write the domain.	x can be anything, so the domain is $x \in \boldsymbol{R}$.
	The graph is exponential, so y values are greater than zero.	y must be positive, so the range is $y = h(x) > 0$
c	Write the operations involved as functions.	$g(x) = x^2$, $f(x) = x + 1$
	Try the composition $f \circ g$.	$f[g(x)] = f(x^2)$ $f(x + 1) = x^2 + 1$ $f[g(x)] = x^2 + 1$
	Do it the other way round as $g \circ f$.	$g[f(x)] = g(x + 1)$ $g(x + 1) = (x + 1)^2$ $g[f(x)] = x^2 + 2x + 1$ As required.
	Now write the domain.	x can be anything, so the domain is $x \in \boldsymbol{R}$.
	The graph is a quadratic, so y values are greater than the minimum value of the function.	y must be positive, so the range is $y = h(x) > 0$

Consider the composition of $f(x) = \sqrt{4 - 4x^2}$ and $g(x) = x^2 - 2x + 5$. $f \circ g$ is not a valid function because the range of g, $y \geq 4$, is outside the domain of f, $x \leq 1$. $g \circ f$ is valid only for the domain of f and the range is the same as the range of g.

IMPORTANT

Functions can only be composed if part of the range of the second function lies within the domain of the first function.

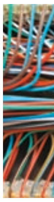
The domain of a composition of functions is either the same as the domain of the second function, or lies within it.

The range of a composition of functions is either the same as the range of the first function, or lies within it.

EXERCISE 3.01 Composition of functions

Concepts and techniques

WS Composition of functions



1 Example 1 Find $g[f(x)]$ given that:

a $g(x) = 2x - 3, f(x) = x - 1$

b $g(x) = x^2 + 2, f(x) = e^x$

c $g(x) = \log_e(x), f(x) = e^{2x}$

d $g(x) = \sin(x), f(x) = 2x$

2 Example 2 Find $f[g(x)]$ for each of the following and compare with $g[f(x)]$ in question **1**.

a $g(x) = 2x - 3, f(x) = x - 1$

b $g(x) = x^2 + 2, f(x) = e^x$

c $g(x) = \log_e(x), f(x) = e^{2x}$

d $g(x) = \sin(x), f(x) = 2x$

3 Example 3 Decompose the following functions into the form $g[f(x)]$.

a $h(x) = x + 3$

b $h(x) = (2x + 5)^3$

c $h(x) = x^2 + 6x + 2$

d $h(x) = e^{x^2}$

e $h(x) = \sin^2(x)$

f $h(x) = \sin(2x)$

Reasoning and communication

4 For the function $h(x) = 4x^2$, write two possible decompositions.

5 For the following functions, find $g \circ f$ and write the domain and range of each.

a $f(x) = 2x, g(x) = \sin(x)$

b $f(x) = -x, g(x) = \log_e(x)$

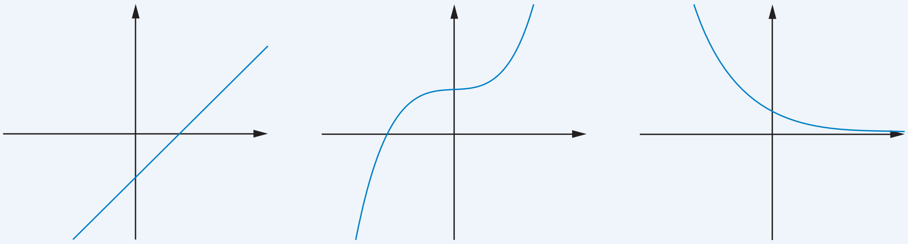
c $f(x) = \dfrac{1}{2x}, g(x) = 2\sin(x)$

6 Write a function that satisfies the property $g \circ f = f \circ g$.

3.02 ONE-TO-ONE FUNCTIONS

INVESTIGATION One-to-one functions

The following functions are one-to-one.



1 Copy each of the graphs above into your book.

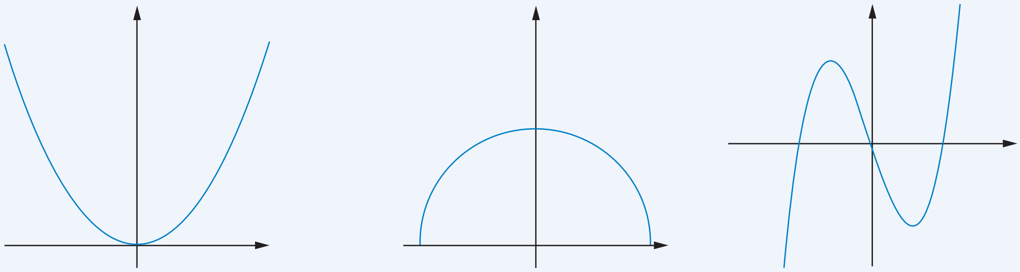
2 Draw a horizontal line across each of these graphs.

3 How many times does the horizontal line pass through the graph?

4 Now draw a vertical line at any point on the graph.

5 How many times does the vertical line pass through the graph?

Look at the following graphs.



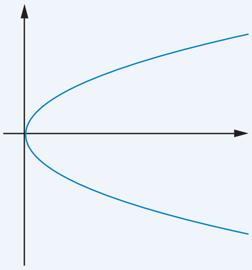
1 How many times does a horizontal line pass through each graph?

2 How many times does a vertical line pass through each graph?

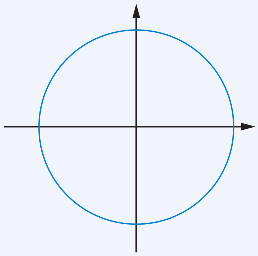
The horizontal and vertical line tests are used to determine if a function is one-to-one or not.

Determine which of the following graphs represent a one-to-one function.

A



B



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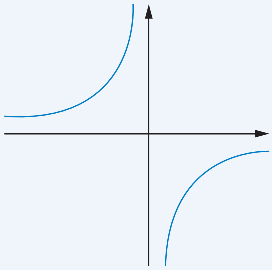

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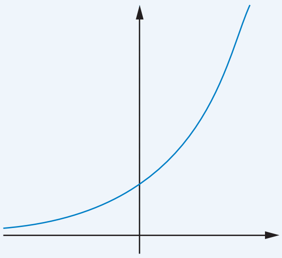
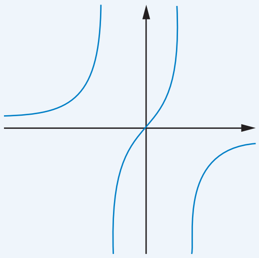
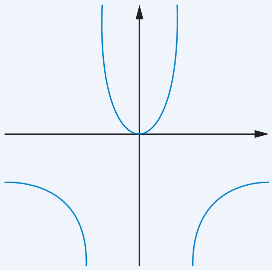


E



F



A function is **one-to-one** if it never assigns two input values to the same output value. In other words, no output value has more than one input value.

For example, to prove that $f(x) = 2x - 5$ is one-to-one, let x and y be integers and suppose that $f(x) = f(y)$. You need to show that $x = y$.

Substituting in our formula for f, $f(x) = f(y)$, so you get $2x - 5 = 2y - 5$, which reduces to $x = y$. Hence, you can say that $f(x) = 2x - 5$ is one-to-one.

To illustrate that not all functions are one-to-one, you can look at the function $f(x) = x^2$. In this case, you get $x^2 = y^2$, which simplifies to $x = \pm y$. Since there is no unique output value, you can say that the function $f(x) = x^2$ is not one-to-one.

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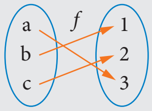
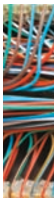
IMPORTANT

A **one-to-one** function satisfies the condition:
If $x \neq y$ then $f(x) \neq f(y)$.

It is sometimes easier to show using the contrapositive:

For all x and y, if $f(x) = f(y)$, then $x = y$.





If $f(x)$ and $g(x)$ are both one-to-one, then $f \circ g$ is one-to-one.

Since $[f \circ g](x) = [f \circ g](y) \rightarrow f[g(x)] = f[g(y)] \rightarrow g(x) = g(y) \rightarrow x = y$.

Example 4

For each of the following functions, determine whether it is one-to-one or not.

a $f(x) = 3x - 4$ **b** $f(x) = x^3$ **c** $f(x) = x^2 - x$

Solution

a Write $f(x) = f(y)$. $3x - 4 = 3y - 4$

Simplify. $3x = 3y$

$x = y$

Determine one-to-oneness. $f(x)$ is one-to-one.

b Write $f(x) = f(y)$. $x^3 = y^3$

Simplify. For $x > 0$, $x = y$

For $x < 0$, $x = y$

Determine one-to-oneness. $f(x)$ is one-to-one.

c Write $f(x) = f(y)$. $x^2 - x = y^2 - y$

Simplify. $x(x - 1) = y(y - 1)$

Determine one-to-oneness. Since $f(0) = 0$ and $f(1) = 0$, $f(x)$ is NOT one-to-one.

It would appear from the earlier investigation that continuous functions that are either increasing or decreasing are one-to-one functions.

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IMPORTANT

A function f is (strictly) **increasing** if for all x and y $(x < y)$, $f(x) < f(y)$.

A function f is (strictly) **decreasing** if for all x and y $(x > y)$, $f(x) > f(y)$.

IMPORTANT

If $f'(x) > 0$ for all x, then $f(x)$ is increasing.

If $f'(x) < 0$ for all x, then $f(x)$ is decreasing.

Continuous increasing functions and continuous decreasing functions are one-to-one.

Example 5

Show that the continuous functions below are increasing, decreasing or neither and hence determine if they are one-to-one.

a $f(x) = -x^5 - 2x^3 + 4$ **b** $f(x) = 2x^3 + 4x + 1$

c $f(x) = 2e^x$ **d** $f(x) = 3x^2 - 12$

Solution

a Find $f'(x)$. $f'(x) = -5x^4 - 6x^2$

Determine if $f(x)$ is increasing or decreasing. $f'(x) = -(5x^4 + 6x^2) < 0$
$f(x)$ is decreasing.

State whether the function is one-to-one or not. Hence, $f(x)$ is one-to-one.

b Find $f'(x)$. $f'(x) = 6x^2 + 4$

Determine if $f(x)$ is increasing or decreasing. $f'(x) > 0$
$f(x)$ is increasing.

State whether the function is one-to-one or not. Hence, $f(x)$ is one-to-one.

c Find $f'(x)$. $f'(x) = 2e^x$

Determine if $f(x)$ is increasing or decreasing. $f'(x) > 0$
$f(x)$ is increasing.

State whether the function is one-to-one or not. Hence, $f(x)$ is one-to-one.

d Find $f'(x)$. $f'(x) = 6x$

Determine if $f(x)$ is increasing or decreasing. $f'(x) < 0, x < 0$ and $f'(x) > 0, x > 0$
$f(x)$ is neither increasing nor decreasing.

State whether the function is one-to-one or not. Hence, $f(x)$ is NOT one-to-one.

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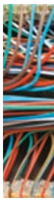
EXERCISE 3.02 One-to-one functions

Concepts and techniques

WS One-to-one functions



1 Example 4 For each of the following functions, determine whether it is one-to-one or not one-to-one.

a $f(x) = x^2 - 4$ b $f(x) = e^{2x} - 1$ c $f(x) = \dfrac{1}{x+3}$ d $f(x) = \dfrac{1}{(x-2)^2}$

2 Show that $f(x) = x^n + x$, $n > 0$ is not one-to-one.

3 Example 5 Show that the following functions are increasing, decreasing or neither and whether they are one-to-one.

a $f(x) = \dfrac{2}{x}$ b $f(x) = x^3 - 9x + 1, x > 2$ c $f(x) = \sin(x)$ d $f(x) = e^{-x}$

4 Show that $f(x) = \pi x - \sin(x)$ is a increasing function for all values of x.

Reasoning and communication

5 Show that $f(x) = 3x^3 - 2$ is one-to-one by decomposing $f(x)$.

6 Using decomposition, show that $h(x) = (\pi - x) - \cos(x)$ is one-to-one.

7 Write an increasing, one-to-one function that can be decomposed to satisfy $f \circ g = g \circ f$.

3.03 INVERSE FUNCTIONS

IMPORTANT

The **inverse** of a relation $R: x \to y$ is the relation $R^{-1}: y \to x$.

If the inverse relation of a function is also a function, it is called the **inverse function**.

A function is an ***n* : 1** (*n*-to-one) relation. This means that every element of the domain maps to only one element in the range. You can also think of this as meaning that every element in the domain belongs to only one ordered pair. For example, $\boldsymbol{y = x^2}$ has pairs such as (3, 9), (–3, 9), (0, 0), and so on.

By contrast, in a **1 : *n*** (one-to-***n***) relation, every element of the range belongs to only one ordered pair. For example, the relation $\boldsymbol{y^2 = x}$ has ordered pairs such as (25, 5), (25, –5), and so on.

As you saw in the previous section, the range and domain elements of a 1 : 1 function are matched, so it is both *n* : 1 and 1 : *n*. This also applies in reverse.

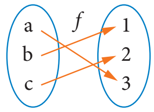
IMPORTANT

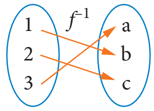
An inverse function exists if and only if the function is one-to-one.

If the function is 1 : 1, then its inverse relation is also 1 : 1, so must be a function.

If the inverse is a function, it must be n : 1, so the function must be 1 : n, so it is both n : 1 and 1 : n, so must be 1 : 1. **QED**

You can sometimes restrict the domain of a function to make it 1 : 1 so that an inverse function will exist. For example, $y = x^2$ is 1 : 1 on the positive real numbers (or on the negative real numbers).





An inverse function is a second function that undoes the work of the first one.

Note: $f^{-1}(x)$ does not mean $\dfrac{1}{f(x)}$.

To work out an inverse function, reverse the operations that f carries out on a number.

For example, let $f(x) = 3x$, so that f takes a number x and multiplies it by 3: $f(x) = 3x$

To define an inverse function that takes 3 times x, and takes it back to x, is the same as saying that $f^{-1}(x)$ divides x by 3. So $f^{-1}(x) = \frac{1}{3}x$ (or $x \div 3$).

Example 6

Write the inverse relation of each of the following functions.

a $y = 2x + 1$ **b** $y = x^3 - 2$ **c** $y = e^{2x}$

Solution

a Write the given function. $y = 2x + 1$

Rearrange to find x as the subject.

$y - 1 = 2x$

$(y - 1) \div 2 = x$

Write the inverse relation. $f^{-1}(x) = \dfrac{x-1}{2}$

Write the given function. $y = x^3 - 2$

Rearrange to find x as the subject.

$y + 2 = x^3$

$\sqrt[3]{y+2} = x$

Write the inverse relation. $f^{-1}(x) = \sqrt[3]{x+2}$

Write the given function. $y = e^{2x}$

Rearrange to find x as the subject.

$\log_e(y) = 2x$

$\dfrac{\log_e(y)}{2} = x$

Write the inverse relation. $f^{-1}(x) = \dfrac{\log_e(x)}{2}$

The inverse relation of $y = f(x)$ can also be found by interchanging the x and y values of the function.

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Example 7

Write the inverse relations of the following.

a $y = 5x - 4$ **b** $y = x^4 + 1$ **c** $y = \pm\sqrt{4-x}$

Solution

a Write the given function. $y = 5x - 4$

Interchange x and y. $x = 5y - 4$

Make y the subject of the equation. $x + 4 = 5y$

$$y = \frac{x+4}{5}$$

Write the inverse relation. $f^{-1}(x) = \frac{x+4}{5}$

b Write the given function. $y = x^4 + 1$

Interchange x and y. $x = y^4 + 1$

Make y the subject of the equation. $x - 1 = y^4$

$$y = \sqrt[4]{x-1}$$

Write the inverse relation. $f^{-1}(x) = \sqrt[4]{x-1}$

c Write the given relation. $y = \pm\sqrt{4-x}$

Interchange x and y. $x = \pm\sqrt{4-y}$

Make y the subject of the equation. $x^2 = 4 - y$

$$y = 4 - x^2$$

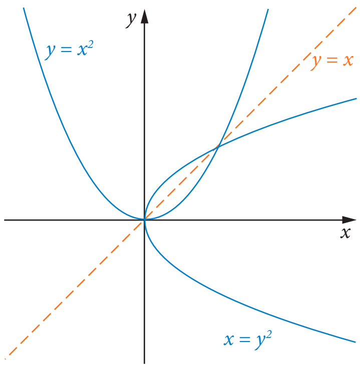
Write the inverse relation. $f^{-1}(x) = 4 - x^2$

On the number plane, the inverse relation can be represented by a reflection of the original function in the line $y = x$.

For example, $y = x^2$ gives $x = y^2$ when reflected in the line $y = x$. Notice that, $x = y^2$ is not a function.

A function has an inverse function if it is one-to-one. That is, it is either increasing or decreasing. Generally, the original function is restricted to include the origin and is either increasing or decreasing.

In the case above, the original function would be restricted to $x \geq 0$ so that the resulting inverse relation only exists in the first quadrant and is thus an inverse function.



IMPORTANT

When an inverse function exists, the domain of the inverse function is the range of the original function, and the range of the inverse function is the domain of the original function.

Example 8

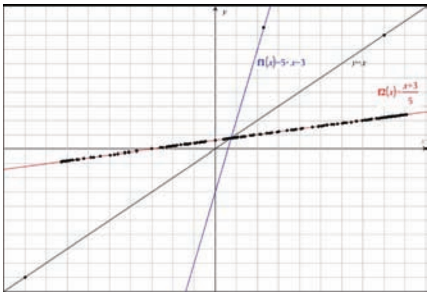
Draw the inverse relations of the following functions using your CAS calculator. Write a restriction, if necessary, to restrict the given function so that the resulting inverse is a function.

a $y = 5x - 3$ **b** $y = x^2 + 3$ **c** $y = e^{2x}$

Solution

a **TI-Nspire CAS**

Graph/Function $y = 5x - 3$.
Draw the line in $y = x$ in **Geometry/Points and lines/Line.**
Draw the inverse by reflection in **Geometry/Transformations/Reflection.**
Turn on the trace in **Trace/Geometry Trace.**

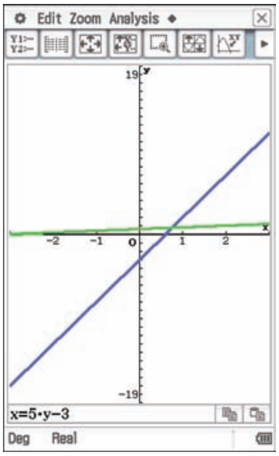


Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

There is no need to limit the domain of the original function as the inverse is a function.

ClassPad

From Menu select Graph&Table. Tap y1 and enter 5 [x] [−] 3 and [EXE]. This will result in $y1 = 5x - 3$. Tap [graph icon] to draw the graph. (This is the blue line. To obtain a useful view of the function and its subsequent inverse, you may need to adjust the domain of the graph by first tapping [view window icon] and changing the maximum and minimum x and y values.)
Tap **Analysis**, then **Sketch** and **Inverse**. The inverse graph is plotted on the screen (here, as a green line). Tap the part of the screen that shows the graph, then Resize to enlarge the graph of both lines so that it uses the whole screen. An expression for the inverse function is shown at the bottom of the screen.



Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

There is no need to limit the domain of the original function as the inverse is a function.

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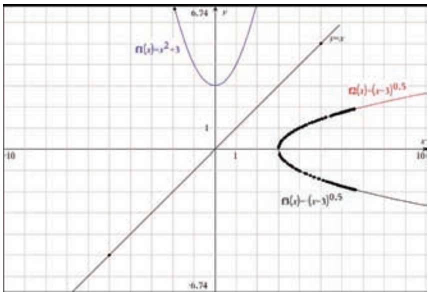
b **TI-Nspire CAS**

Graph/Function $y = x^2 + 3$.
Draw the line in $y = x$ in **Geometry/Points and lines/Line**.
Draw the inverse by reflection in **Geometry/Transformations/Reflection**.
Turn on the trace in **Trace/Geometry Trace**.

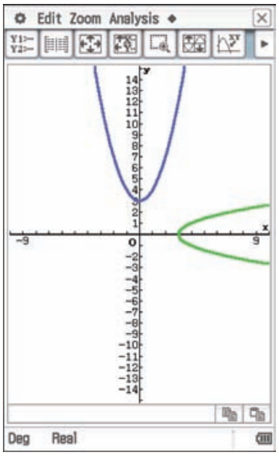


Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

The resulting inverse is not a function so the original function needs to be limited.
Some possible limited domains are $1 \le x \le 5$, $-4 \le x \le 0$, $-4 < x < 0$, $x \le 0$ or $x \ge 0$.

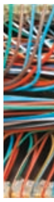
ClassPad

Proceed in a similar manner as for part **a**.
At $y1$ enter x [^] 2 [+] 3 and press [EXE].
Tap [graph icon] to draw the graph. Tap **Analysis**, then **Sketch** and **Inverse**. The inverse graph is plotted on the screen (here, as a green line).



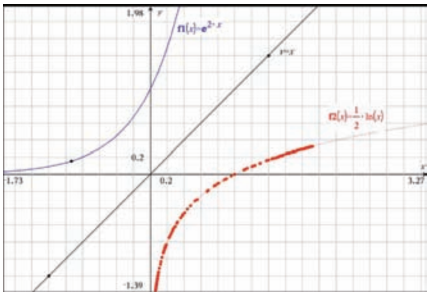
Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

The resulting inverse is not a function so the original function needs to be limited.
Some possible limited domains are $1 \le x \le 5$, $-4 \le x \le 0$, $-4 < x < 0$, $x \le 0$ or $x \ge 0$.



c **TI-Nspire CAS**

Graph/Function $y = e^{2x}$.
Draw the line in $y = x$ in **Geometry/Points and lines/Line**.
Draw the inverse by reflection in **Geometry/Transformations/Reflection**.
Turn on the trace in **Trace/Geometry Trace**.

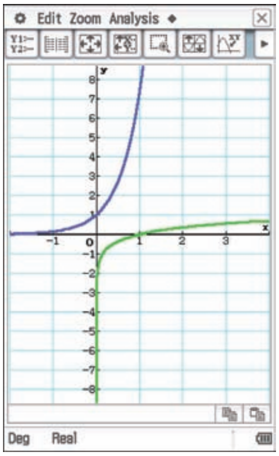


Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

There is no need to limit the domain of the original function as the inverse is a function.

ClassPad

Proceeding in a similar manner as before, tap $y1$ and enter e^{2x}. (The exponential is obtained by first tapping [Keyboard], then either [Math1] or [Math2], then [$e^{\blacksquare}$].) Tap [graph icon] to draw the graph. Tap **Analysis**, then **Sketch** and **Inverse**. The inverse graph is plotted on the screen (here, as a green line).



Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

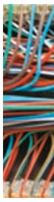
There is no need to limit the domain of the original function as the inverse is a function.

You can restrict the domain of any function and it will still be a function. However, you usually choose the largest possible continuous domain as 'the' domain of a function. In parts **a** and **c** of Example **8**, the domains do not *need* to be restricted to make the inverse a function. In part **b**, the domain of the function does need to be restricted to make the inverse a function. The largest possible continuous domains you could choose are $x \leq 0$ or $x \geq 0$. Given a choice between a positive and a negative domain, most people choose the positive one, so it is standard practice to choose the domain $x \geq 0$ for part **b**.

EXERCISE 3.03 Inverse functions

Concepts and techniques

Inverse functions



1 Example 6 Write the inverse relation of each the following functions by making y the subject, that is, $x = f(y)$.

a $y = 2x - 1$ b $y = x^2 + 1$ c $y = 2\log_e(x - 1)$

d $y = 1 + \sqrt{x}$ e $y = x^3 + 1$

2 Determine which of the inverse relations in question **1** are functions.

3 What is the condition for the function $y = \frac{1}{x^2}$ to have an inverse function?

4 Example 7 Write the inverse relation of each the following functions by interchanging x and y and finding $y = f(x)$.

a $y = x^3$ b $y = 3x - 2$ c $y = e^x$

d $y = \frac{2}{x}$ e $y = \frac{1}{1 - x}$

5 Restrict the domain of each of the following functions to a strictly increasing curve, find the inverse function, and state the domain and range of the inverse function.

a $y = x^2$ b $y = 3x^2 - 1$ c $y = (x - 2)^4$

d $y = \frac{3}{x^2}$ e $y = \frac{2}{x^4}$

6 Example 8 Sketch the following functions and their inverse relation by reflecting each in the line $y = x$.

a $y = 2x^2$ b $y = x^2 + 2$ c $y = (x - 3)^2$

d $y = x^2 - 2x$ e $y = x^4 - 1$

7 By restricting the following functions to (i) strictly increasing and (ii) strictly decreasing curves, write down the inverse function.

a $f(x) = x^2 - 2x$ b $y = x^4 - 2$ c $y = 2x^4 + 1$

d $f(x) = x^2 - 6x + 1$ e $y = x^2 + 4x - 3$

Reasoning and communication

8 a Write down the domain and range of the function $y = \frac{1}{x - 2}$.

b Find the inverse function.

c Find the domain and range of the inverse function.

9 By restricting the domain of $f(x) = x^2 - 4x$, find the inverse function defined over that domain.

10 Find where the function $y = x^2 + 6x - 1$ is strictly decreasing and state the inverse function over that domain.

3.04 ABSOLUTE VALUE FUNCTIONS

The **absolute value**, also known as **modulus**, of a number $x \in \mathbf{R}$ is written as $|x|$ and it is given by the non-negative number that defines its magnitude. For example, $|7| = 7 = |-7|$. In practical terms, the absolute value of a number is its distance from the origin.

IMPORTANT

The **absolute value** of x is given by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Note that the effect of taking the absolute value of a number is to remove the minus sign if the number is negative and to leave the number unchanged if it is non-negative.

Example 9

Evaluate each of the following.

a $|6|$ b $|-16|$ c $|4-6|$

d $|3-6|-|6-3|$ e $|-6|\times(-4)$

Solution

a Write the question.
Use the definition to determine the answer.

$|6|$
$= 6$, since $6 > 0$.

b Write the question.
Use the definition to determine the answer.

$|-16|$
$= -(-16)$
$= 16$, since $-16 < 0$.

c Write the question.
Use the definition to determine the answer.

$|4-6|$
$= |-2|$
$= -(-2)$
$= 2$, since $-2 < 0$.

d Write the question.
Use the definition to determine the answer.

$|3-6|-|6-3|$
$= |-3|-|3|$
$= -(-3)-3$
$= 0$, since $-3 < 0$ and $3 > 0$.

e Write the question.
Use the definition to determine the answer.

$|-6|\times(-4)$
$= -(-6)\times(-4)$
$= -24$, since $-6 < 0$.

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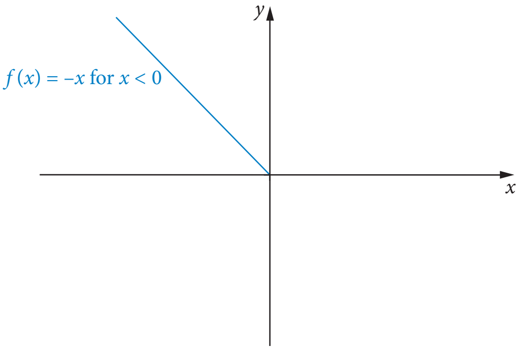
The **absolute value function**, $y = |x|$ is a basic function. By definition, the function is represented by:

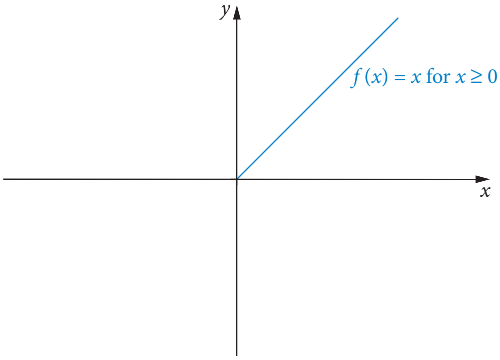
Absolute values

$$y = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

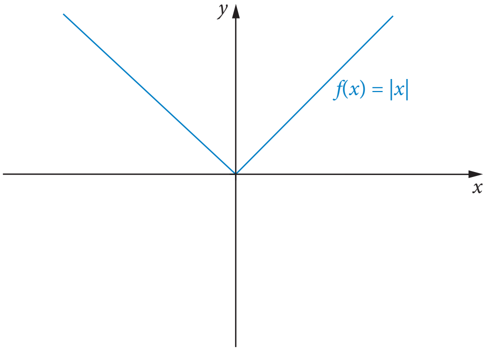
This function can be thought of as two components put together to form one function. It is the combination of $y = x$ (when $x \geq 0$) and $y = -x$ (when $x < 0$).

Graphically you get:





Combining over the respective domains you get:



Of course, you can easily sketch these functions using a table of values.

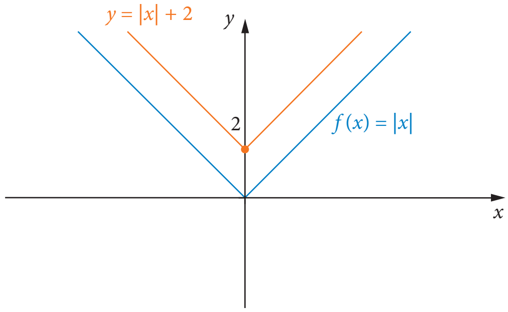
Example 10

Sketch the following absolute value functions and state their domain and range.

a $y = |x| + 2$ **b** $y = |x| - 2$ **c** $y = \frac{1}{2}|x| + 1$ **d** $y = |x - 3|$

Solution

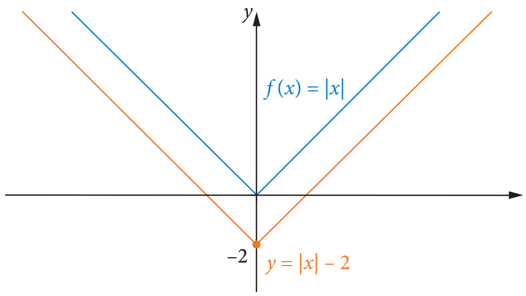
a Sketch $f(x) = |x|$.
Add 2 to each ordinate (y-value).



State the domain and range.

Domain: $x \in \boldsymbol{R}$, range: $y \geq 2$

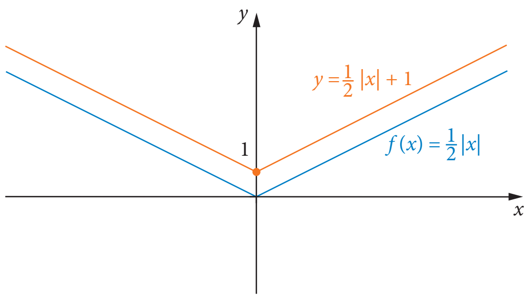
b Sketch $f(x) = |x|$.
Subtract 2 from each ordinate (y-value).



State the domain and range.

Domain: $x \in \boldsymbol{R}$, range: $y \geq -2$

c Sketch $f(x) = \frac{1}{2}|x|$.
Add 1 to each ordinate (y-value).



State the domain and range.

Domain: $x \in \boldsymbol{R}$, range: $y \geq 1$

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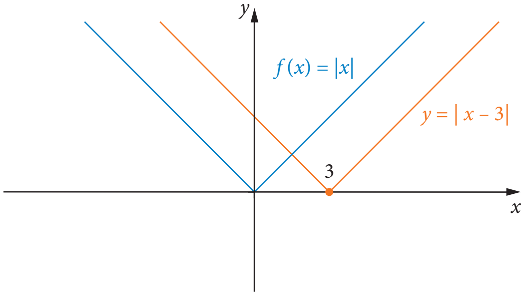
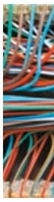
d Sketch $f(x) = |x|$.
Move each abscissa (x-value) 3 units to the right.



State the domain and range.

Domain: $x \in \boldsymbol{R}$, range: $y \geq 0$



IMPORTANT

To sketch $y = |f(x)|$, sketch $y = f(x)$ and make all negative y values positive. That is, the negative parts of the graph are reflected in the x-axis.

In sketching absolute value functions, you can solve equations like $|x - 3| = 5$ and inequations like $|2x + 3| \geq 5$ by observing where solutions lie on the number line.

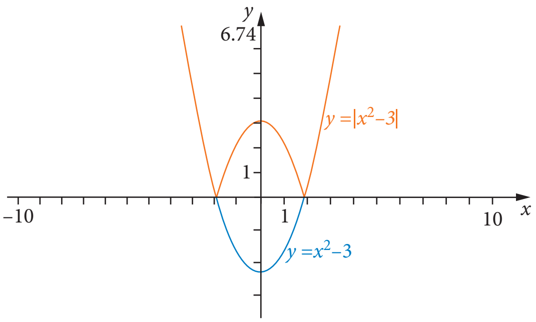
Example 11

Sketch the following absolute value functions and state their domain and range.

a $y = |x^2 - 3|$ **b** $y = |x(x-2)(x+2)|$ **c** $y = |\sin(x)|$ **d** $y = \left|\frac{1}{x}\right|$

Solution

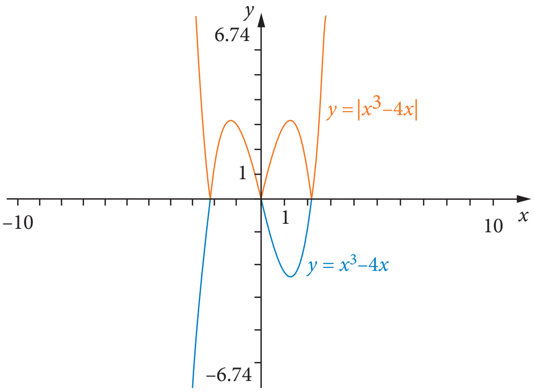
a Sketch $y = x^2 - 3$.
Reflect the negative parts so that they are positive.



State the domain and range.

Domain: $x \in \boldsymbol{R}$, range: $y \geq 0$.

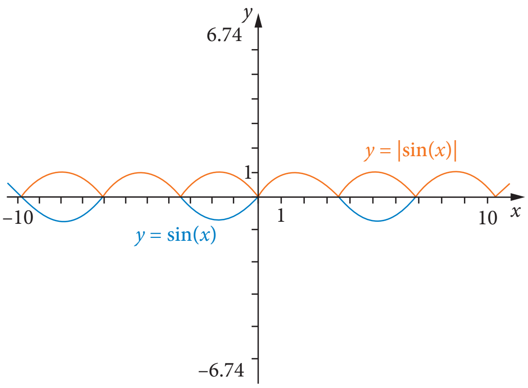
b Sketch $y = x(x - 2)(x + 2)$.
Reflect the negative parts so that they are positive.



State the domain and range.

Domain: $x \in \boldsymbol{R}$, range: $y \geq 0$.

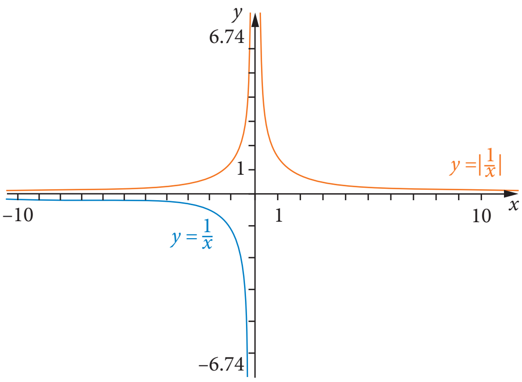
c Sketch $y = \sin(x)$.
Reflect the negative parts so that they are positive.



State the domain and range.

Domain: $x \in \boldsymbol{R}$, range: $0 \leq y \leq 1$.

d Sketch $y = \frac{1}{x}$.
Reflect the negative parts so that they are positive.



State the domain and range.

Domain: $x \in \boldsymbol{R}$, $x \neq 0$, range: $y > 0$.

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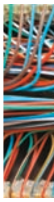
EXERCISE 3.04 Absolute value functions

Concepts and techniques



1 Example 9 Evaluate each of the following.

a $|-4|$
b $|-4+3|$
c $|-4|-3|-4|-3$
d $|-4\times 3|$
e $-|3|^2$
f $|-4\times 3|-2|4-3|$
g $|-6\times 2|-|-6-6|$
h $|-5\times 2|\div|5+5|$
i $|-6\div 3|-|-4-3|$
j $|-5\times(-2)|+|-6+5|$

2 Solve the following equations.

a $|x-3|=5$
b $|2x+3|=5$
c $-2|x-3|=8$
d $3|x+1|=12$
e $\left|\frac{1}{x}\right|=4$

3 Solve $|x-3|=|2x-1|$. Remember to check your solutions or use a graphics calculator to assist you with this one.

4 Example 10 Sketch the following absolute value functions and state their domain and range.

a $y=|x+4|$
b $y=|x|+4$
c $y=2|x+2|$
d $y=2|x-2|$
e $y=|x-2|-2$
f $y=-|x|+2$
g $y=-|x+2|$
h $y=-2|x-2|-1$

5 Solve the equation $|x+2|=3$ by sketching the graphs of $y=|x+2|$ and $y=3$.

6 Solve the equation $|x+2|=|3-x|$ by sketching the appropriate graphs.

7 Example 11 Sketch the following absolute value functions and state their domain and range.

a $y=|\cos(x)|$
b $y=|e^x|$
c $y=|x^2-1|$
d $y=-|x^2-4|$
e $y=\frac{1}{|x+1|}$
f $y=|(x-1)(x+2)(x-3)|$

Reasoning and communication

8 Solve the inequality $|(x+2)(x-2)|\geq 4$

9 Solve the inequality $\frac{1}{|x-1|}\geq 3$

10 Solve the inequality $|x-2|+|x+2|<5$

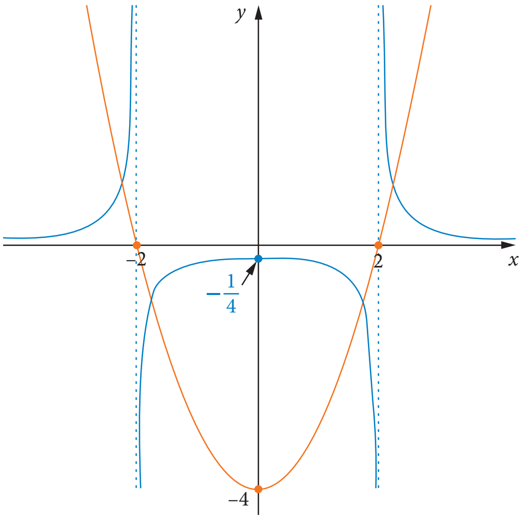
3.05 GRAPHS OF $y = f(x)$, $y = \frac{1}{f(x)}$, $y = f(|x|)$

A **reciprocal function** has the form $y = \frac{1}{f(x)}$, where $f(x)$ is a polynomial and $f(x) \neq 0$.

The graphs below show the function $f(x) = x^2 - 4$ on the same plane as its reciprocal function $y = \frac{1}{x^2 - 4}$.

You can see that:

- when the original function passes through the x-axis, the reciprocal function has an asymptote
- as the original function gets large, its reciprocal function gets small (approaching zero)



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There are other features concerning reciprocal functions that are worth noting and a summary of these follows below.

IMPORTANT

For the function $y = f(x)$, the following features will assist in sketching $y = \frac{1}{f(x)}$.

- $f(x)$ and $\frac{1}{f(x)}$ have the same sign
- $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect where $f(x) = \pm 1$
- Zeros of $y = f(x)$ correspond to vertical asymptotes of $y = \frac{1}{f(x)}$
- As $f(x)$ increases, $\frac{1}{f(x)}$ decreases and vice versa
- As $f(x) \to \pm\infty$, $\frac{1}{f(x)} \to 0$ and as $f(x) \to 0$, $\frac{1}{f(x)} \to \pm\infty$
- Maximum turning points of $y = f(x)$ correspond to minimum turning points of $y = \frac{1}{f(x)}$ and vice versa

Example 12

Sketch the following graphs and their reciprocal functions.

a $f(x) = x - 2$ **b** $f(x) = (x - 2)(x + 1)$ **c** $f(x) = x(x - 3)(x + 3)$

Solution

a Sketch $f(x) = x - 2$

Determine the y-intercept for $f(x) = x - 2$ and take the reciprocal for the y-intercept of $f(x) = \frac{1}{x - 2}$.

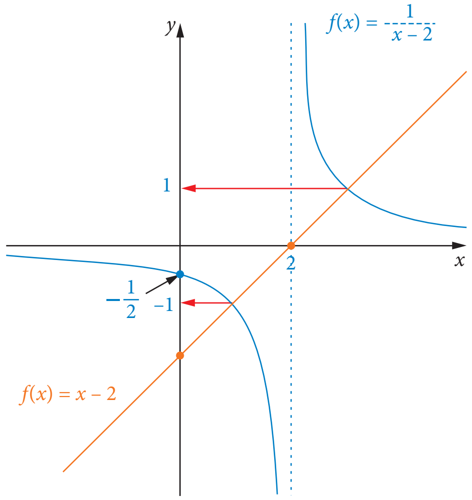
Identify where $y = 0$ and sketch an asymptote.

Identify where $y = 1$ and mark this shared point.

Identify where $y = -1$ and mark this shared point.

Check that where $f(x) > 0$ that $\frac{1}{f(x)} > 0$ and that where $f(x) < 0$ that $\frac{1}{f(x)} < 0$.

Sketch the graph of $f(x) = \frac{1}{x - 2}$.



b Sketch $f(x) = (x - 2)(x + 1)$

Determine the y-intercept and take the reciprocal for the y-intercept of

$f(x) = \dfrac{1}{(x-2)(x+1)}$.

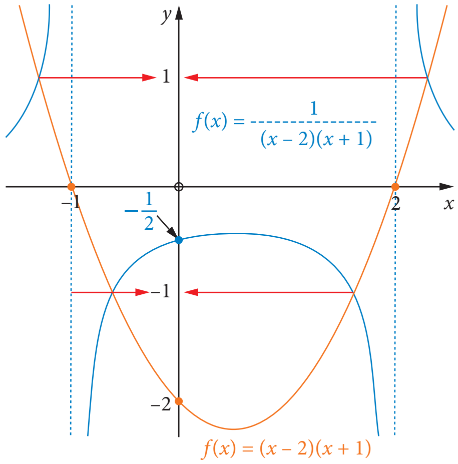
Identify where $y = 0$ and add an asymptote.

Identify where $y = 1$ and mark this shared point.

Identify where $y = -1$ and mark this shared point.

Check that where $f(x) > 0$ that $\dfrac{1}{f(x)} > 0$ and that where $f(x) < 0$ that $\dfrac{1}{f(x)} < 0$.

Sketch the graph of $f(x) = \dfrac{1}{(x-2)(x+1)}$.



c Sketch $f(x) = x(x - 3)(x + 3)$

Determine the y-intercept and take the reciprocal for the y-intercept of

$f(x) = \dfrac{1}{x(x-3)(x+3)}$.

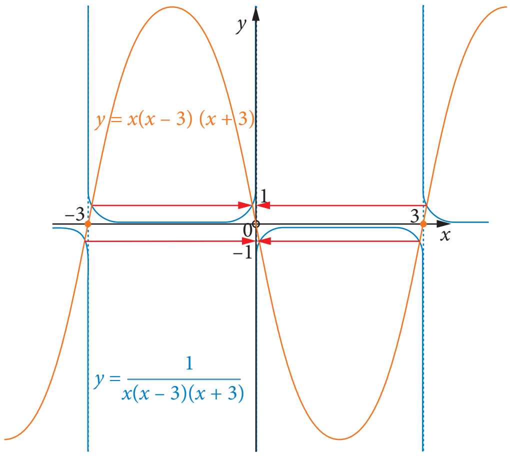
Identify where $y = 0$ and add an asymptote.

Identify where $y = 1$ and mark this shared point.

Identify where $y = -1$ and mark this shared point.

Check that where $f(x) > 0$ that $\dfrac{1}{f(x)} > 0$ and that where $f(x) < 0$ that $\dfrac{1}{f(x)} < 0$.

Sketch the graph of $f(x) = \dfrac{1}{x(x-3)(x+3)}$.



Sketching the function $f(x) = (x - 4)(x + 2)$ below on the same plane as $f(x) = (|x| - 4)(|x| + 2)$, you can see that when $x \geq 0$ the graphs are the same and that when $x < 0$ the graph of $y = f(|x|)$ is the reflection of $y = f(x)$ in the y-axis.

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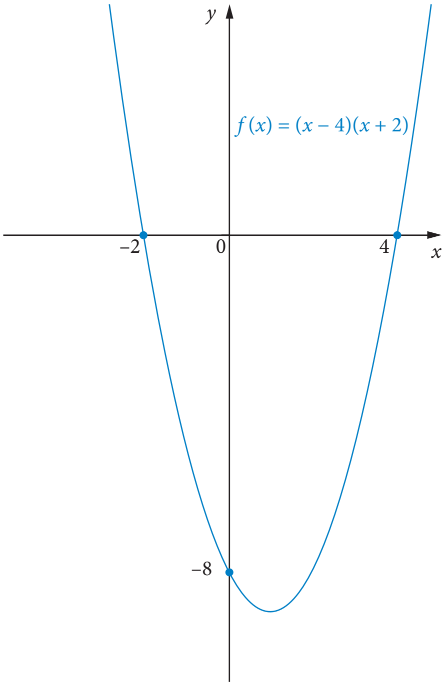
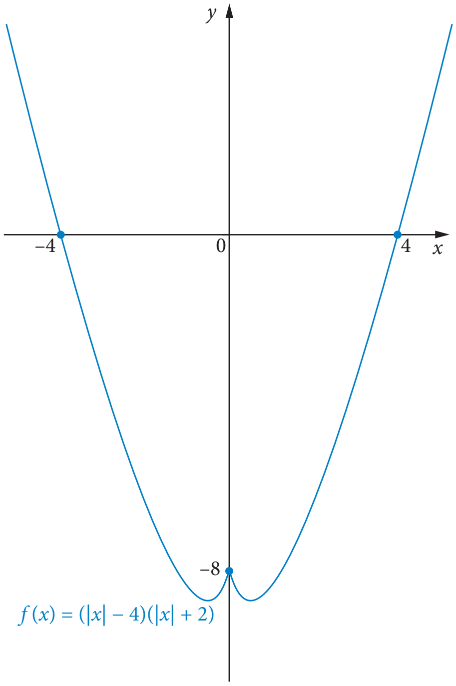




Given the graph $y = f(x)$, to sketch the graph of $y = f(|x|)$, sketch the graph of $y = f(x)$ for $x \geq 0$ only and then reflect this graph in the y-axis to complete it.

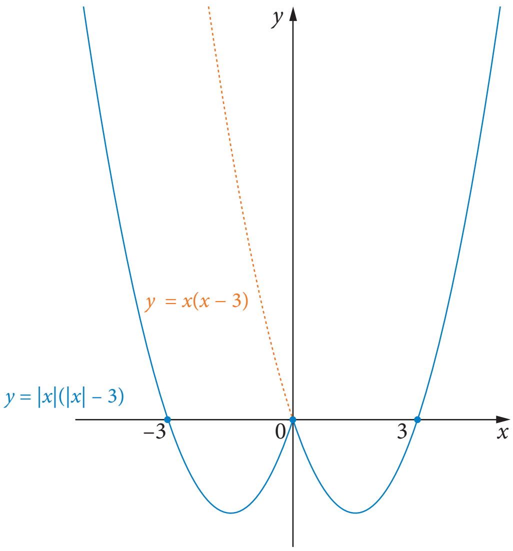
Example 13

Sketch the graphs of $y = f(x)$ and $y = f(|x|)$ for the following equations.

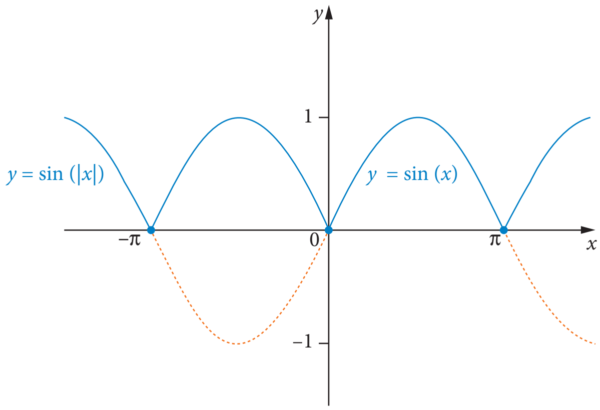
a $y = x(x - 3)$ **b** $y = \sin(x)$ **c** $y = e^x$

Solution

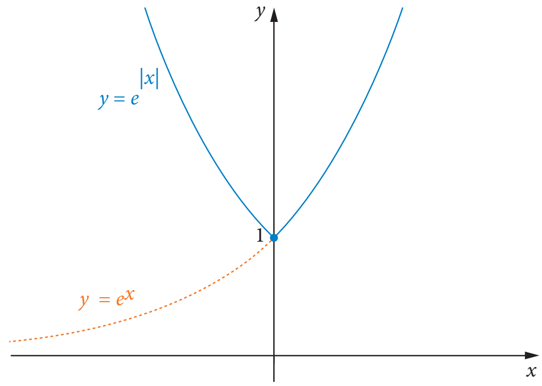
a Sketch $y = x(x - 3)$ for $x \geq 0$ in orange.
Reflect this graph in the y-axis to get the required graph in blue.
Label the graph.



b Sketch $y = \sin(x)$ for $x \geq 0$ in orange.
Reflect this graph in the y-axis to get the required graph in blue.
Label the graph.



c Sketch $y = e^x$ for $x \geq 0$ in orange.
Reflect this graph in the y-axis to get the required graph in blue.
Label the graph.



EXERCISE 3.05 Graphs of $y = f(x)$, $y = \frac{1}{f(x)}$, $y = f(|x|)$

Concepts and techniques

WS
Transformations of absolute values and hyperbolas

1 Example 12 Sketch the following graphs and their reciprocal functions.

a $f(x) = x + 2$
b $f(x) = (x - 3)(x + 2)$
c $f(x) = x(x - 2)(x + 3)$
d $f(x) = e^x$
e $f(x) = \sin(x)$

2 Solve the inequation $\frac{1}{2x - 3} \geq 1$ by first sketching $y = 2x - 3$.

3 Solve the inequation $\frac{1}{(x - 3)(x + 3)} > 0$ by first graphing $y = x^2 - 9$.

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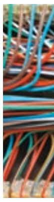
4 Example 13 Sketch the graphs of $y = f(|x|)$ for the following graphs.

a $f(x) = x^2 + 2$
b $f(x) = \log_e(x)$
c $f(x) = 1 + x$
d $f(x) = 4x - x^2$
e $f(x) = x^2(x - 2)$

5 Solve the inequation $6|x| - x^2 = 9$ by first sketching $y = 6|x| - x^2$.

6 Solve the inequation $\log_e(|x|) < 2$ by first sketching $y = \log_e(x)$.



Reasoning and communication

7 Sketch the graph of $y = \dfrac{1}{1+|x|}$.

8 By first sketching $y = x^2 + 1$, sketch the graph of $y = \dfrac{1}{x^2+1}$, then $y = \dfrac{x}{x^2+1}$ and hence $y = \left|\dfrac{x}{x^2+1}\right|$.

9 Using a graph of $y = x(x - 2)(x + 4)$, solve $\dfrac{1}{x(x-2)(x+4)} < 0$

3.06 GRAPHS OF RATIONAL FUNCTIONS

A reciprocal function has a denominator that is another function, but the numerator is 1. If the numerator and denominator are both polynomials, then the function is called a **rational function**. A reciprocal function is a special case of a rational function.

IMPORTANT

A **rational function** is one that can be expressed in the form $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x)$ is not the zero polynomial.

$y = \dfrac{x^2}{1+x}$, $y = \dfrac{x^2-1}{x^2+1}$ and $y = \dfrac{x^3+2x^2+x+2}{x+2}$ are all rational functions.

The first step in sketching simple rational functions is to sketch the polynomial in the denominator and then the reciprocal of this function.

Rational functions that have a constant as the numerator are the simplest ones.

Example 14

Sketch the following rational functions.

a $f(x)=\dfrac{-1}{(x-1)(x+3)}$ **b** $f(x)=\dfrac{10}{x(x+3)(x-3)}$

Solution

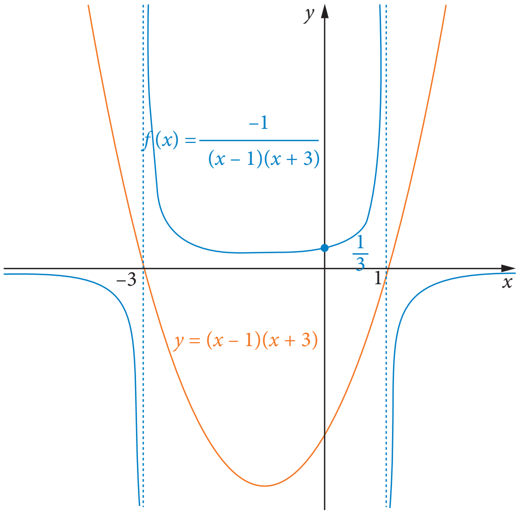
a Sketch the polynomial $y=(x-1)(x+3)$. Determine any intercepts and the asymptotes of $f(x)$.

$f(0)=\frac{1}{3}$.

When $x=-3, 1, f(x)\to\pm\infty$ (asymptotes).

The desired function has the *opposite* signs to $y=(x-1)(x+3)$. The negative sign makes it the reflection in the x-axis of the reciprocal function.

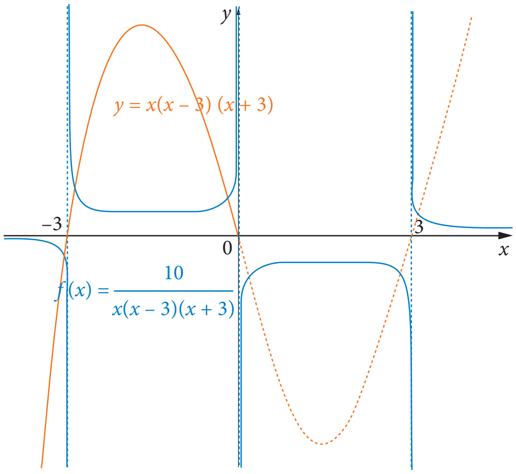
Sketch the rational function $f(x)=\dfrac{-1}{(x-1)(x+3)}$ using points and features.



b Sketch the polynomial $y=x(x+3)(x-3)$. Determine any intercepts and asymptotes of the reciprocal function. The desired function is stretched by the factor 10 in the vertical direction from the reciprocal function.

When $x=0, \pm 3, f(x)\to\pm\infty$ (asymptotes).

Sketch the rational function $f(x)=\dfrac{10}{x(x-3)(x+3)}$ using points and features.



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Polynomial functions are continuous, so rational functions are also continuous, except for the asymptotes. This means that rational functions can change sign only at zeros and asymptotes.

Consider $f(x) = \dfrac{x^2 - 3x + 5}{x^3 + x + 1}$, where the degree of the numerator is one less than the degree of the denominator. You can write the reciprocal as $\dfrac{1}{f(x)} = x - 3 - \dfrac{13x - 16}{x^2 - 3x + 5}$.

As $x \to \pm\infty$, $\dfrac{1}{f(x)}$ becomes very large, so $f(x) \to 0$. The same reasoning applies to any rational function whose numerator is of lower degree than the denominator.

IMPORTANT

A rational function whose numerator is of lower degree than the denominator has a horizontal asymptote of $x = 0$. You can sketch rational functions of this type using the horizontal asymptote for behaviour as $x \to \pm\infty$ and considering the sign of the function between zeroes and asymptotes.

Example 15

Sketch the following rational functions.

a $y = \dfrac{x}{(x+3)(x-3)}$ **b** $f(x) = \dfrac{x-1}{(x+2)(x-4)}$ **c** $y = \dfrac{(x+1)(x-2)}{(x+2)(x-1)(x-4)}$

Solution

a The numerator is of lower degree than the denominator, so the horizontal asymptote is $x = 0$.

Determine any zeros and asymptotes.

There is a zero at $x = 0$ and asymptotes at $x = -3$ and $x = 3$. The y-intercept is the zero.

Work out the sign of the function between the zero and asymptotes.

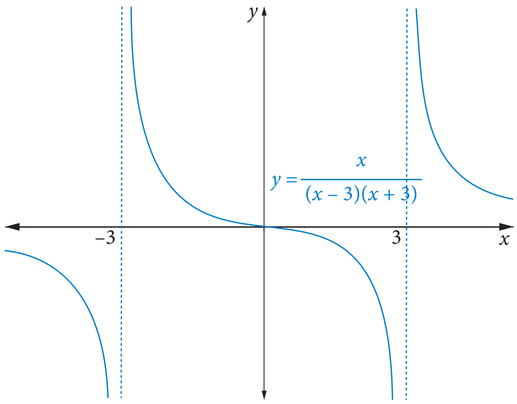
For $x < -3$; $y < 0$

For $-3 < x < 0$; $y > 0$

For $0 < x < 3$; $y < 0$

For $x > 3$; $y > 0$

Sketch the rational function $y = \dfrac{x}{(x-3)(x+3)}$.



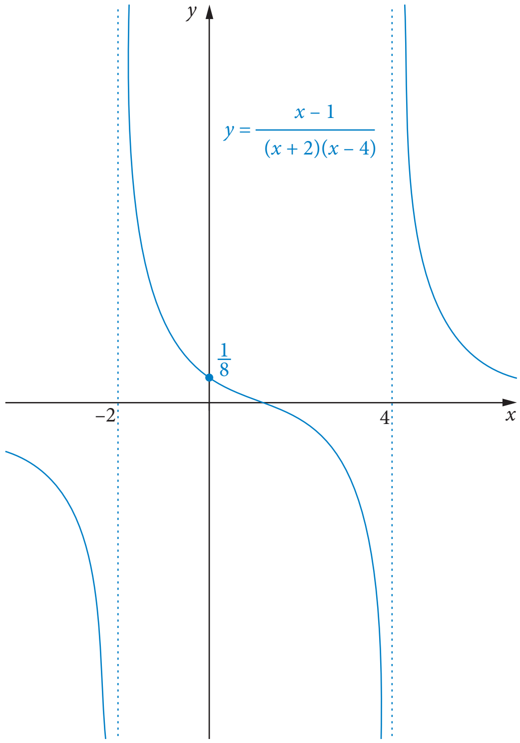
b The numerator is of lower degree than the denominator, so the horizontal asymptote is $x = 0$. Determine any zeros and asymptotes.

There is a zero at $x = 1$ and asymptotes at $x = -2$ and $x = 4$. The y-intercept is $\frac{1}{8}$.

Work out the sign of the function between the zero and asymptotes.

For $x < -2$; $y < 0$
For $-2 < x < 1$; $y > 0$
For $1 < x < 4$; $y < 0$
For $x > 4$; $y > 0$

Sketch the rational function $y = \dfrac{x-1}{(x+2)(x-4)}$.



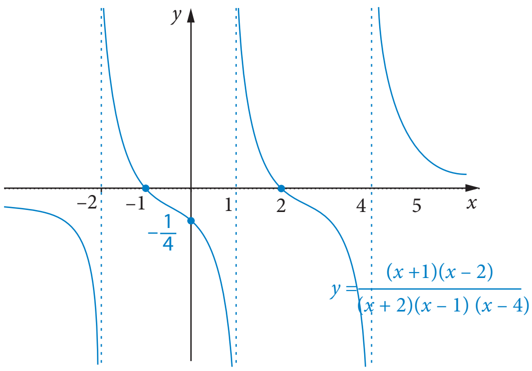
c The numerator is of lower degree than the denominator, so the horizontal asymptote is $x = 0$. Determine any zeros and asymptotes.
There are zeros at $x = -1$ and $x = 2$.
The asymptotes are at $x = -2$, $x = 1$ and $x = 4$.
The y-intercept is $-\frac{1}{4}$.

Work out the sign of the function between the zero and asymptotes.

For $x < -2$; $y < 0$
For $-2 < x < -1$; $y > 0$
For $1 < x < 1$; $y < 0$
For $1 < x < 2$, $y > 0$
For $1 < x < 2$; $y < 0$
For $x > 4$; $y > 0$

Sketch the rational function

$$y = \frac{x^2}{(x+2)(x-1)(x-4)}.$$



The degree of the numerator of $f(x) = \dfrac{x^2 - 4x + 4}{x-1}$ is greater than the degree of the denominator.

You can change this to functions that you know how to sketch by dividing the numerator by the denominator.

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$$
\begin{array}{r}
x - 3 \\
x - 1 \overline{\big) \; x^2 - 4x + 4} \\
\underline{x^2 - x } \\
-3x + 4 \\
\underline{-3x + 3} \\
1
\end{array}
$$

Thus $x^2 - 4x + 4 = (x - 1)(x - 3) + 1$ and so $f(x) = \dfrac{x^2-4x+4}{x-1} = x-3+\dfrac{1}{x-1}$.

This function has an asymptote at $x = 1$ and as $x \to \pm\infty$, $\dfrac{1}{x-1}$ becomes very small, so $f(x)$ behaves like $y = x - 3$, which is a straight line. This is called a **sloping asymptote**.

If you divide out $f(x)=\dfrac{3x+4}{x+2}$, you get $f(x)=3-\dfrac{2}{x+2}$, which behaves like $y = 3$ as $x \to \pm\infty$. This is a **horizontal axis**.

IMPORTANT

You need to divide a rational function whose numerator has degree greater than or equal to the degree of the denominator. This will give the non-vertical asymptote as the dividend. You can then find the vertical asymptotes, zeros and y-intercept and sketch the function by considering the sign of the function between zeros and asymptotes.

Example 16

Sketch the following rational functions.

a $y=\dfrac{x+1}{x-1}$ **b** $f(x)=\dfrac{2x^2}{x+2}$ **c** $g(x)=\dfrac{x^2-x+1}{x-1}$

Solution

a Dividing through gives

$x + 1 = (x - 1) \times 1 + 2$.

This gives $y=1+\dfrac{2}{x-1}$.

There is a horizontal asymptote at $y = 1$.

There is a vertical asymptote at $x = 1$.

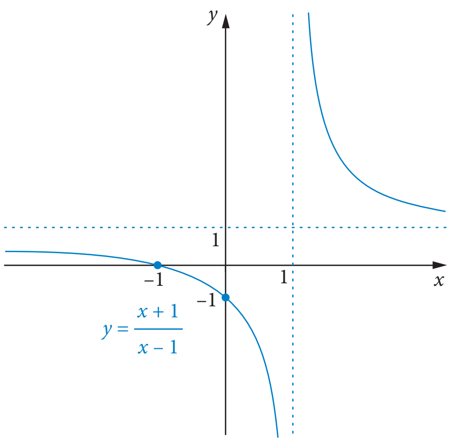
The zero is at $x = -1$.

The y-intercept is at -1.

Consider the sign of y.

For $x < -1$;	$y > 0$
For $1 < x < 1$,	$y < 0$
For $x > 1$,	$y > 0$

Sketch the function.



b Dividing through gives
$2x^2 = (x+2) \times (2x-4) + 8$.

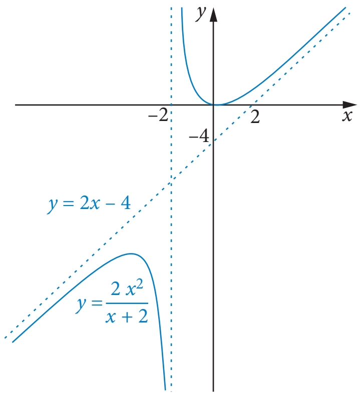
This gives $y = 2x - 4 + \dfrac{8}{x+2}$.

This means that $y = 2x - 4$ is a sloping asymptote.

There is a vertical asymptote at $x = -2$.
There is a double zero at $x = 0$.
Consider the sign of y.

For $x < -2$;	$y < 0$
For $-2 < x < 0$,	$y > 0$
For $x > 0$,	$y > 0$

Sketch the function.

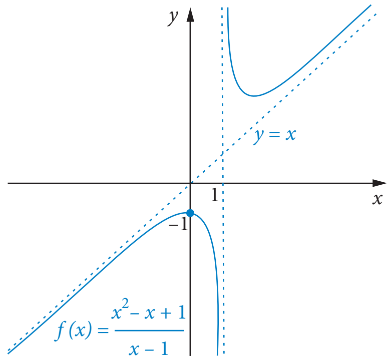


c Dividing through gives
$x^2 - x + 1 = x(x-1) + 1$

Thus, $g(x) = x + \dfrac{1}{x-1}$.
$y = x$ is a sloping asymptote.
There is a vertical asymptote at $x = 1$.
$x^2 - x + 1$ has no real zeros, so neither does $g(x)$.
The y-intercept is at -1.
Consider the sign of y.

For $x < 1$;	$y < 0$
For $x > 1$,	$y > 0$

Sketch the function.



IMPORTANT

To find the equations of the horizontal (or otherwise) asymptotes for a rational function, you examine the degrees of the polynomials in the numerator $P(x)$ and the denominator $Q(x)$.

- If deg $P(x) <$ deg $Q(x)$, then there is a horizontal asymptote at $y = 0$.
- If deg $P(x) =$ deg $Q(x)$, then there is a horizontal asymptote at $y = \dfrac{a}{b}$ where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.
- If deg $P(x) >$ deg $Q(x)$, then the equation of the asymptote will not be horizontal, it will be a slanted line if the difference in degrees is one, and parabolic if the difference in degrees is two.

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EXERCISE 3.06 Graphs of rational functions

Concepts and techniques

1 Example 14 Sketch each of the following rational functions, showing any asymptotes and intercepts.

a $y = \dfrac{1}{x+3}$ b $y = \dfrac{1}{(x+3)(x-1)}$ c $y = \dfrac{-1}{x^2-1}$

d $y = \dfrac{2}{x^2+x-6}$ e $y = \dfrac{-2}{x^2+7x+6}$ f $y = \dfrac{-1}{x(x+1)(x-2)}$

2 Example 15 Sketch each of the following rational functions, showing any asymptotes and intercepts.

a $y = \dfrac{x+1}{x^2}$ b $y = \dfrac{x+1}{x^2-1}$ c $y = \dfrac{x+2}{(x-1)(x-4)}$

d $y = \dfrac{x}{(x-1)(x+1)}$ e $y = \dfrac{x^2+1}{x(x-1)(x+1)}$ f $y = \dfrac{x^2+2x+1}{x(x^2-1)}$

3 Example 16 Sketch each of the following rational functions, showing any asymptotes and intercepts.

a $y = \dfrac{x+1}{x-1}$ b $y = \dfrac{x-1}{x+1}$ c $y = \dfrac{x^2}{x-1}$ d $y = \dfrac{x^3-8}{(x-1)(x+1)}$

Reasoning and communication

4 Using the graph of the rational function $y = \dfrac{1}{(x-2)(x+4)}$, find the solution of the inequality $\dfrac{1}{(x-2)(x+4)} > 0$.

5 Solve the inequality $\dfrac{x-2}{x^2-3x-4} \le 0$.

6 Solve the inequality $\dfrac{x^2-2}{x^2-2x+1} \le 0$.

7 The value of a motorcycle (V), t years after it was bought from a store, can be estimated by the rational function $V(t) = \dfrac{(6+t)}{0.002t}$, for $t \ge 1$.

a What is the value of the motorcycle after 2 years?

b What is the value of the motorcycle after 10 years?

c Sketch a graph of the value of the motorcycle for the first ten years.

iStockphoto/Martin Wimmer

8 The annual sales (in billions of dollars) of computer games from 1990 to 2010 can be modelled by $G(t)=\frac{850t^2+3200}{110t^2+1000}$, $0 \le t \le 20$, where t is the number of years since 1990.

a For which year were the total sales of computer games about \$5.5 billion?

b If the sales of these computer games continued each year, what limit do these sales approach?

c Sketch a graph of this model for the sales of computer games.

Shutterstock.com/Jacek Chabraszewski

9 Stephen has made 60 out of his 80 free throw attempts at basketball this season.

a Write a rational expression that represents Stephen's free throw percentage if he makes his next x free throw attempts.

b How many consecutive free throws must Stephen make in order to raise his free throw percentage to at least 90%?

c Sketch a graph to show Stephen's rate of success with free throws this season if he makes all his free throws for the rest of the season.

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CHAPTER SUMMARY
FUNCTIONS AND SKETCHING GRAPHS

3

- The **composite** function $g \circ f$ is the function given by $g[f(x)] = gf(x)$.

  This is also called a function of a function and sometimes written as gf.
- In general, $g[f(x)]$ is not equal to $f[g(x)]$.
- Functions can only be composed if part of the range of the second function lies within the domain of the first function.
- The domain of a composed function is either the same as the domain of the second function, or lies inside it.
- The range of a composed function is either the same as the range of the first function, or lies inside it.
- A **one-to-one** function satisfies the condition:

  $x \neq y \Rightarrow f(x) \neq f(y)$.
  - The contrapositive, $f(x) = f(y) \Rightarrow x = y$, may be easier to use.
- A function f is (strictly) **increasing** if for all x and y $(x < y)$, $f(x) < f(y)$.
- A function f is (strictly) **decreasing** if for all x and y $(x > y)$, $f(x) > f(y)$.
- If $f'(x) > 0$ for all x, then $f(x)$ is increasing.
- If $f'(x) < 0$ for all x, then $f(x)$ is decreasing.
- Continuous increasing functions and continuous decreasing functions are one-to-one.
- The **inverse** of a relation $R: x \to y$ is the relation $R^{-1}: y \to x$.
- If the inverse relation of a function f is also a function, it is called the **inverse function**, written as f^{-1}.
- An inverse function exists if and only if the function is one-to-one.
- The inverse relation of $y = f(x)$ can be found by making x the subject of the function. You can also interchange x and y first and then find y.
- On the number plane, the inverse relation can be represented by a reflection of the original function in the line $y = x$.
- When an inverse function exists, the domain of the inverse function is the range of the original function, and the range of the inverse function is the domain of the original function.
- The absolute value, also known as modulus, of a number $x \in \boldsymbol{R}$ is written as $|x|$ and is given by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

- For the function $y = f(x)$, the following features will assist in sketching $y = \frac{1}{f(x)}$.
  - $f(x)$ and $\frac{1}{f(x)}$ have the same sign
  - $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect, where $f(x) = \pm 1$
  - the zeros of $y = f(x)$ correspond to vertical asymptotes of $y = \frac{1}{f(x)}$
  - As $f(x)$ increases, $\frac{1}{f(x)}$ decreases and visa versa
  - As $f(x) \to \pm\infty$, $\frac{1}{f(x)} \to 0$ and as $f(x) \to 0$, $\frac{1}{f(x)} \to \pm\infty$
  - Maximum turning points of $y = f(x)$ correspond to minimum turning points of $y = \frac{1}{f(x)}$ and vice versa
- A rational function is one that can be expressed in the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x)$ is not the zero polynomial.
- For a rational function with numerator $P(x)$ and denominator $Q(x)$:
  - if deg $P(x) <$ deg $Q(x)$, then there is a horizontal asymptote at $y = 0$
  - if deg $P(x) =$ deg $Q(x)$, then there is a horizontal asymptote at $y = \frac{a}{b}$, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator
  - if deg $P(x) >$ deg $Q(x)$, there may be a slanted straight line or curved asymptote. The asymptote is the dividend when the numerator is divided by the denominator.
  - you can sketch a rational function by finding the non-vertical asymptote as above, the y-intercept, the vertical asymptotes and zeros. Then use the sign between the zeros and vertical asymptotes to assist in sketching the function.

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CHAPTER REVIEW
FUNCTIONS AND SKETCHING GRAPHS

3

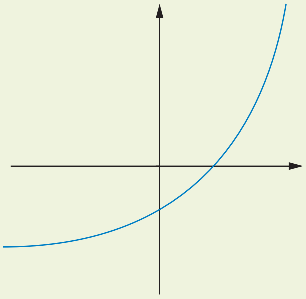
Multiple choice

1 Example 1 Which of the following gives $f[g(x)]$, where $f(x) = x^2 - 1$ and $g(x) = x + 1$?

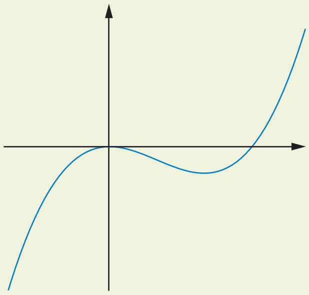
A $(x^2 - 1)(x + 1)$ B x^2 C $x^2 + 1$ D $x^2 + 2x$ E $x^2 + 2x + 1$

2 Example 4 Which of the following is NOT a function?

A



B



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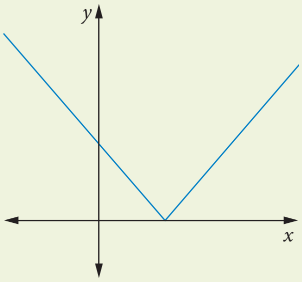
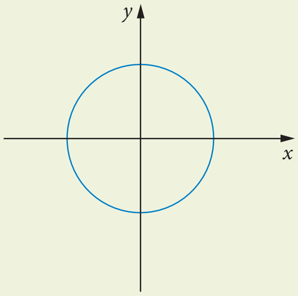
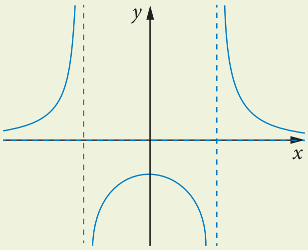


D



E



3 Example 2 Find $[f \circ g](x)$ given that $f(x) = 2e^x$ and $g(x) = \log_e(\frac{1}{2}x)$.

A $\log_e(e^x)$ B x C e^x D $2\log_e(x)$ E $\frac{1}{2}e^x$

4 Example 3 Which of the following pairs of functions result when the function $h(x) = x^2 + 2x + 2$ is decomposed?

A $f(x) = x^2, g(x) = 2x + 2$ B $f(x) = x^2 + 2x, g(x) = 2$ C $f(x) = x^2 + 1, g(x) = x + 1$
D $f(x) = x^2 + x, g(x) = x + 2$ E $f(x) = x^2 - 1, g(x) = x - 1$

5 Example 5 Determine which of the following functions is strictly decreasing.

A $f(x) = e^x$ B $f(x) = x^2 + 2x + 1$ C $f(x) = \frac{1}{x+1}$
D $f(x) = \sin(x)$ E $f(x) = \frac{x}{\log_e(x)}$

Short answer

6 Example 6 Find the inverse relation of $y = \frac{2}{x-3}$.

7 Example 7 Find the inverse relation of $y = \sqrt{5-x}$.

8 Example 8 Sketch the graph of $y = 2e^{3x}$ and its inverse function.

9 Example 9 Evaluate the following.

a $|-4| - 6$ b $|6 \times (-2)| + 1$ c $|3 \times (-2)| - |6 \times (-3)|$

10 Example 10 Sketch the function $y = -|x| - 3$ and state its domain and range.

11 Example 11 Solve the equation $|x+1| = -\frac{1}{2}x + 1$ using graphs of $y = |x + 1|$ and $y = -\frac{1}{2}x + 1$.

12 Example 11 Sketch the function $y = |(x - 1)(x + 2)(2 - x)|$. Make the function $g(x) = (x - 1)(x + 2)(2 - x)$.

13 Example 12 Sketch the reciprocal function $f(x)$ of $g(x) = (x-1)(x+2)(2-x)$.

14 Example 13 Sketch the function $f(|x|)$ given that $f(x) = (x + 3)(x + 1)(x - 2)$.

15 Example 14 Sketch the function $f(x) = \frac{1}{(x-2)(x+1)}$.

16 Example 15 Sketch the function $f(x) = \frac{x}{x^2 - 4}$.

17 Example 16 Sketch the function $f(x) = \frac{x^2}{x^2 - 4}$.

Application

18 Write an equation for a rational function that has asymptotes $y = x$ and $x = -2$, and which passes through (0, 2).

19 Solve the inequality $\frac{x^2}{x^2 - 4} \leq 2$.

20 So far this year, Ross has completed 15 out of 17 homework assignments in Mathematics. Ross intends to submit all assignments from now on. Write an expression for his percentage assignment submission rate and solve using graphing or algebra to find how many consecutive assignments he must now submit to raise his rate to 95%.

Qz Practice quiz

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